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# Path-ordered Phase Factors in Scalar Quantum Electrodynamics

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## Abstract

Starting from linear equations for the complex scalar field, the two- and three-point Green's functions are obtained in the infrared approximation. We show that the infrared singularity factorizes in the vertex function as in spinorial QED, reproducing in a straightforward way the result of lenghty perturbative calculations.

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## 1. Introduction

The major difficulty in dealing with long-range potentials in the relativistic quantum theory is the factorization of infrared divergences out of scattering amplitudes, in order to get finite cross-sections for the corresponding processes. Who first provided a deeper insight to this problem in spinorial quantum electrodynamics were Bloch and Nordsieck[1], over half a century ago. Later, it was shown[2],[3] that if we cast bremsstrahlung contributions to all orders in perturbation theory, we get finite cross-sections. Pursuing the Bloch and Nordsieck lines, Chung[4], Kulish and Faddeev[5], Zwanziger[6], and others work in the construction of a new space of asymptotic states,

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containing an infinite number of coherent photons, in order to extract finite elements from the S-matrix. Nevertheless, despite these great efforts, one feels that few progress have been made along the last decades at a more operational level, and even more puzzling is the case of non-Abelian gauge theories, such as quantum chromodynamics, and quantum gravity. In the early 80's, applying the concept of path-ordered phase factors, exhaustive explored in QCD, Kubo[7] reproduced the Kulish-Faddeev and Grammer-Yennie results for spinorial QED in the infrared asymptotic region. The power of his method lies on the simplicity of calculations. Furthermore, it can be extended to the determination of generic n-point Green's functions. It would be desirable that Kubo's method also applies to scalar QED in order to obtain information about its behavior at the infrared, since few emphasis has been given to this matter and the analysis is much less obvious in this case. As we shall see, it is sufficient to write the interaction Hamiltonian of scalar QED in a suitable form.

Keeping this in mind, we will adopt an alternative approach to scalar QED, which is also useful for the investigation of electromagnetic properties of particles of either zero or unity spin. It is specially adequate to the study of the renormalizability of scalar QED as an effective theory[8], equivalent to the original, whose structure is similar to that of spinorial QED. This formalism is briefly described in section 2. In section 3 we give the asymptotic form for the current of charged mesons. In the sequence, sections 4 and 5, we evaluate the dressed meson propagator and the vertex function in the infrared region and confront the results with those obtained through different methods.

## 2. The Duffin-Kemmer equation for the scalar field

For our purposes it is convenient to start not with the usual Klein-Gordon field equation but with the first order equation

$$(i\beta_\mu \partial^\mu - m)\psi(x) = 0 \ , \tag{1}$$

known as the Duffin-Kemmer equation[9], which has the same appearance as the Dirac equation, where $\beta_\mu$ are matrices satisfying the algebra

$$\beta_\mu \beta_\nu \beta_\lambda + \beta_\lambda \beta_\nu \beta_\mu = g_{\mu\nu}\beta_\lambda + g_{\lambda\nu}\beta_\mu \ , \tag{2}$$

with metric tensor $g_{\mu\nu} = diag(1, -1, -1, -1)$. In the linear equation (1) the field $\psi$ has five components, the scalar field and its four derivatives with respect to the coordinates and the time which in turn transform like the components of a vector.

The adjoint $\overline{\psi}(x)$ of $\psi(x)$ is defined by

$$\overline{\psi}(x) = \psi^\dagger(x)(2{\beta_0}^2 - I) \ , \tag{3}$$

where $I$ is the five-rowed identity matrix. Using that the matrix $\beta_0$ is Hermitian and the matrices $\beta_k$ are anti-Hermitian for $k = 1, 2, 3$, it can be shown that $\overline{\psi}(x)$ satisfies

$$i\partial^\mu \overline{\psi}(x)\beta_\mu + m\overline{\psi}(x) = 0 \ . \tag{4}$$

Equations (1) and (4) are obtained from the Lagrangian density

$$\mathcal{L}_0^m = \frac{i}{2}(\overline{\psi}\beta_\mu \partial^\mu \psi - \partial^\mu \overline{\psi}\beta_\mu \psi) - m\overline{\psi}\psi \ , \tag{5}$$

and the corresponding charge-current vector for the scalar field can be written as

$$j_\mu = e\overline{\psi}\beta_\mu \psi \ . \tag{6}$$

The sistem of the meson field and the electromagnetic field in interaction is classically described by the Lagrangian density

$$\mathcal{L} = \mathcal{L}_0^m + \mathcal{L}_0^{em} + \mathcal{L}_{int} \ , \tag{7}$$

where $\mathcal{L}_0^{em}$ represents the Lagrangian density for the free electromagnetic field and

$$\mathcal{L}_{int} = j_\mu A^\mu \tag{8}$$

the interaction Lagrangian density. Thus, the Heisenberg equations of motion for interacting fields are given by

$$[i\beta^\mu(\partial_\mu - ieA_\mu(x)) - m]\psi(x) = 0 \ , \tag{9}$$

$$\overline{\psi}(x)[i\beta^\mu(\overleftarrow{\partial}_\mu - ieA_\mu(x)) + m] = 0 \ , \tag{10}$$

$$\Box A_\mu(x) = j_\mu(x) \ . \tag{11}$$

The operator $\overleftarrow{\partial}_\mu$ in equation (10) acts to the left by definition.

In spinor quantum electrodynamics the field operators $\psi(x)$ and $A_\mu(x)$ in the interaction picture have the same form as the operators for the corresponding free fields. The situation is different here, since not all the components of $\psi(x)$ are dynamically independent. This can be seen if we multiply Eq.(9) to the left by $I - \beta_0^2$ and use relation (2), arriving at

$$(I - \beta_0^2)\psi(x) = -\frac{i}{m}\vec{\beta}.(\overrightarrow{\nabla} - ie\vec{A}(x))\beta_0^2\psi(x) \ . \tag{12}$$

In view of this subsidiary condition for the field operators $\psi(x)$, the interaction Hamiltonian density turns out to be

$$\mathcal{H}_{int}(x) = -e\overline{\psi}^{(0)}(x)\beta^\mu A_\mu^{(0)}(x)[I + \frac{e}{m}(I - \beta_0^2)\beta^\nu A_\nu^{(0)}(x)]\psi^{(0)}(x) \ , \tag{13}$$

where $\psi^{(0)}(x)$ and $A_\mu^{(0)}$ are the operators for the free meson and electromagnetic fields, respectively.

The vacuum expectation value of the chronological product of $\psi^{(0)}(x)$ and $\overline{\psi}^{(0)}(y)$ is

$$< 0|T\{\psi^{(0)}(x)\overline{\psi}^{(0)}(y)\}|0 >= G^c(x-y) + \frac{1}{m}(I - \beta_0^2)\delta^{(4)}(x-y) \ , \tag{14}$$

where $G^c(x-y)$ is the Green's function for the free meson field equation, namely

$$G^c(x-y) = \frac{1}{(2\pi)^4}\int d^4p\, e^{-ip(x-y)}S_F(p) \ , \tag{15}$$

$$S_F(p) = (\not{p} - m)^{-1} = \frac{\not{p}(\not{p}+m) + p^2 - m^2}{m(p^2 - m^2 + i\epsilon)} \ , \tag{16}$$

with $\not{p} \equiv \beta_\mu p^\mu$.

We note that the Hamiltonian density (13) contains a term of order $e^2$. Therefore, the S-matrix expansion in terms of $\mathcal{H}(x)$ will not be an expansion in powers of $e$. Fortunately, the second term of (14) gives a contribution to the S-matrix which just cancel all effects caused by the term of order $e^2$ in the interaction Hamiltonian density[8]. As a result we can evaluate S-matrix elements from the effective Hamiltonian

$$\mathcal{H}_{int}^{eff}(x) = -j_\mu^{(0)}(x)A^{\mu(0)}(x) \ , \tag{17}$$

$$j_\mu^{(0)}(x) = e\overline{\psi}^{(0)}(x)\beta_\mu\psi^{(0)}(x) \ , \tag{18}$$

with effective pairing between meson field operators which differs from (14) by the absence of the last term.

## 3. The infrared approximation

We may express the free field operators $A_\mu^{(0)}$, $\psi^{(0)}$ and $\overline{\psi}^{(0)}$ in terms of creation and annihilation operators:

$$A_\mu^{(0)}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{d^3\vec{k}}{\sqrt{2k_0}}[a_\mu(\vec{k})e^{-ikx} + a_\mu^\dagger(\vec{k})e^{ikx}] \ , \tag{19}$$

$$\psi^{(0)}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3\vec{p}\sqrt{\frac{m}{2p_0}}[b(\vec{p})u(\vec{p})e^{-ipx} + d^\dagger(\vec{p})v(\vec{p})e^{ipx}] \ , \tag{20}$$

$$\overline{\psi}^{(0)}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3\vec{p'}\sqrt{\frac{m}{2p'_0}}[d(\vec{p'})\overline{v}(\vec{p'})e^{-ip'x} + b^\dagger(\vec{p'})\overline{u}(\vec{p'})e^{ip'x}] \ . \tag{21}$$

In order to obtain an expression for the interaction Hamiltonian

$$H_{int} = -e\int d^3\vec{x} : \overline{\psi}^{(0)}(x)\beta_\mu\psi^{(0)}(x) : A^{\mu(0)}(x) \tag{22}$$

in the interaction picture, it's sufficient to substitute formulas (19)-(21) into the above equation. The resulting expression is an integral over the momenta $\vec{p}$, $\vec{p'}$ and $\vec{k}$ of mesons and photons, which are related by $\vec{p'} = \vec{p} + \vec{k}$.

In the neighborhood of small momentum $k_\mu$ we have

$$p_0 - p'_0 \mp k_0 \sim \mp\frac{p.k}{p_0} \ , \tag{23}$$

and

$$\overline{u}(\vec{p'})\beta_\mu u(\vec{p}) = \overline{v}(\vec{p'})\beta_\mu v(\vec{p}) \sim \frac{p_\mu}{m} \ . \tag{24}$$

The last result follows from an identity which can be deduced in much the same way as the Gordon identity, using the algebra for three beta matrices, and Dirac-like orthonormality conditions. Thus, in the infrared asymptotic region the interaction Hamiltonian turns out to be

$$H^{as}_{int} = -\frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3\vec{p}\, \frac{p_\mu}{p_0} \rho(\vec{p}) \int \frac{d^3\vec{k}}{\sqrt{2k_0}} [a_\mu(\vec{k}) e^{-i(pk/p_0)t} + a^\dagger_\mu(\vec{k}) e^{i(pk/p_0)t}] \ , \tag{25}$$

where

$$\rho(\vec{p}) = e[b^\dagger(\vec{p})b(\vec{p}) - d^\dagger(\vec{p})d(\vec{p})] \tag{26}$$

is the charge-density operator, which clearly satisfies the commutation relations

$$[b(\vec{q})\, ,\, \rho(\vec{p})] = b(\vec{q})\delta^{(3)}(\vec{q}-\vec{p}) \ , \tag{27}$$

$$[d^\dagger(\vec{q})\, ,\, \rho(\vec{p})] = d^\dagger(\vec{q})\delta^{(3)}(\vec{q}-\vec{p}) \ . \tag{28}$$

By requiring that the asymptotic interaction Hamiltonian density be of the form (17), we obtain from (19) and (25)

$$j^{as}_\mu(x) = \int d^3\vec{p}\, \frac{p_\mu}{p_0} \rho(\vec{p}) \delta^{(3)}(\vec{x} - \frac{\vec{p}}{p_0} t) \ , \tag{29}$$

for the asymptotic charged-meson current operator in the interaction picture. This corresponds to the current of a point charge with momentum $p_\mu$ in a classical uniform motion with velocity $p_\mu/p_0$, if we consider charged-particle states. This coincides with the Kulish-Faddeev result[5] for fermions, thus reinforcing their statement that expression (29) is quite general and applies in the case of charged particles with arbitrary spin.

## 4. Infrared Phase Factors

We now apply Kubo's technique[7] to extract infrared singularities from Green's functions in the effective scalar meson theory described in section 2.

So, let us first consider the simplest case of the two-point Green's function $G^c(x-y)$ given by (15). In the interaction picture, this can be written as

$$G^c(x-y) = \frac{<0|T\{\psi^{(0)}(x)\overline{\psi}^{(0)}(y)\mathcal{S}\}|0>}{<0|\mathcal{S}|0>} \ , \tag{30}$$

where $\psi^{(0)}$ and $\overline{\psi}^{(0)}$ are defined in (20) and (21), respectively, with

$$\mathcal{S} = T\{\exp[i\int d^4x' \, j^{(0)}_\mu(x') A^{\mu(0)}(x')]\} \ . \tag{31}$$

As we are mostly interested in the infrared asymptotic behavior of the theory, we replace $j^{(0)}_\mu$ in (31) by expression (29) for $j^{as}_\mu$ and neglect contributions from vacuum polarization graphs, like in the Bloch-Nordsieck model[1], since there are no antiparticles in the limit of low frequencies. As a result the two-point Green's function can be brought into the form

$$G^c(x-y) =< 0|T\{\psi^{(0)}(x)\exp[i\int_{y_0}^{x_0} d^4x' \, j^{as}_\mu(x')A^{\mu(0)}(x')]\overline{\psi}^{(0)}(y)\}|0> \ , \tag{32}$$

where we have set the vacuum expectation value of the S-matrix equal to unity. Taking the commutation relations (27) and (28) into account, it follows from (15) that the Fourier transform of $G^c(x-y)$ is equal to

$$G^c(p) =< 0|T\{\exp[ie\int_y^x dx'_\mu \, A^{\mu(0)}(x')]\}|0> S_F(p) \ , \tag{33}$$

where $S_F(p)$ is the Feynman propagator for charged mesons, given by (16), and

$$x_\mu \equiv \frac{p_\mu}{p_0} x_0 \ \ , \ \ y_\mu \equiv \frac{p_\mu}{p_0} y_0 \ . \tag{34}$$

Expanding the exponential and using Wick's theorem, the path-ordered phase factor in (33) becomes

$$\begin{aligned} &< 0|T\{\exp[ie\int_y^x dx'_\mu \, A^{\mu(0)}(x')]\}|0> \\ &\qquad = \exp\{\frac{ie^2}{2}\int_y^x dx'_\mu \int_y^x dx''_\nu D^{\mu\nu}_c(x'-x'')\} \ , \end{aligned} \tag{35}$$

where

$$\begin{aligned} D^{\mu\nu}_c(x'-x'') &= i < 0|T\{A^\mu(x')A^\nu(x'')\}|0> \\ &= \frac{1}{(2\pi)^n}\int d^nk \, e^{-ik(x'-x'')} D^{\mu\nu}_F(k) \ , \end{aligned} \tag{36}$$

with

$$D_F^{\mu\nu}(k) = \frac{g^{\mu\nu}}{k^2 + i\epsilon} - (1-a)\frac{k_\mu k_\nu}{(k^2+i\epsilon)^2} \ , \tag{37}$$

$a$ being the gauge parameter. We leave open the dimension in the Fourier transform (36) to advance that we are going to employ dimensional regularization for divergent integrals to appear in the next steps of calculation. Inserting (36) into expression (35) and carrying out the integrations with respect to $x'_\mu$ and $x''_\nu$, we are left with

$$< 0|T\{\exp[ie\int_y^x dx'_\mu \, A^{\mu(0)}(x')]\}|0 >$$

$$= \exp\{\frac{-ie^2}{2}\int \frac{d^nk}{(2\pi)^n}\frac{1}{(k.p)^2}[e^{i(\frac{x_0-y_0}{p_0})k.p} + e^{-i(\frac{x_0-y_0}{p_0})k.p} - 2] \, p_\mu D_F^{\mu\nu}(k) p_\nu \ . \tag{38}$$

It is more convenient to rewrite the above equation in the form

$$< 0|T\{\exp[ie\int_y^x dx'_\mu \, A^{\mu(0)}(x')]\}|0 >= e^{-f(\frac{x_0-y_0}{p_0})} \ , \tag{39}$$

where

$$f(\nu) = -ie^2\int \frac{d^nk}{(2\pi)^n}\, p_\alpha D_F^{\alpha\beta}(k) p_\beta \int_0^\nu d\nu' \int_0^{\nu'} d\nu'' \, e^{i\nu''(k.p)} \ . \tag{40}$$

The last expression also appears in the calculation of the Green's function in the Bloch-Nordsieck model for spinorial QED[10],[11]. Here we only transcribe the final result for $f(\nu)$, after performing the k-integration:

$$f(\nu) = (-i)^n \frac{e^2}{8\pi^{n/2}}\frac{\Gamma(n/2-1)}{3-n}[2p^2 + (1-a)(n-3)]P(\nu) \ , \tag{41}$$

with

$$P(\nu) \equiv -(-i\epsilon)^{3-n}\nu + \frac{1}{4-n}[(\nu - i\epsilon)^{4-n} - (-i\epsilon)^{4-n}] \ . \tag{42}$$

For $n = 4 - \eta$, we obtain in the limit $\eta \to 0$

$$f(\nu) = -\frac{e^2}{8\pi^2}(2p^2 + 1 - a)\{-i\frac{\nu}{\epsilon} + \log[\frac{(i\nu+\epsilon)}{\epsilon}]\} \ . \tag{43}$$

We now investigate the infrared singularity wich occurs in the unrenormalized vertex function $\mathcal{B}_\mu(p,p')$, by considering the three-point Green's function

$$G_\mu(x,y,z)=\frac{1}{(2\pi)^8}\int d^4p\int d^4p' e^{-ip(x-y)-ip'(y-z)}G_\mu(p,p') \qquad (44)$$

in the interaction picture. It can be shown in the same grounds as above that

$$G_\mu(p,p')=S_F(p)\beta_\mu S_F(p')<0|T\{\exp[ie\int_z^x dx'_\mu A^{\mu(0)}(x')]\}|0> , \qquad (45)$$

where

$$x_\mu\equiv\frac{p_\mu}{p_0}x_0 \ \ , \ \ z_\mu\equiv\frac{p'_\mu}{p'_0}z_0 \ , \qquad (46)$$

and the integration path is along the trajectory described by the charged meson. For convenience, we take the path from z to y and then, from y to x, and put $y=0$. Thus, the path-ordered phase factor in (45) is found to be

$$\begin{aligned}&<0|T\{\exp[ie\int_z^x dx'_\mu A^{\mu(0)}(x')]\}|0>\\&=\exp\{\frac{-ie^2}{2}[\int_0^x dx'_\mu\int_0^x dx''_\nu D_F^{\mu\nu}(x'-x'')+\int_z^0 dx'_\mu\int_z^0 dx''_\nu D_F^{\mu\nu}(x'-x'')]\\&\qquad -ie^2\int_0^x dx'_\mu\int_z^0 dx''_\nu D_F^{\mu\nu}(x'-x'')\} , \end{aligned}\qquad (47)$$

where $x$ and $z$ are given by (46).

The first and second terms in the argument of the exponential in the last equation can be cast into $G^c(p)$ and $G^c(p')$, respectively, while the third one together with $\beta_\mu$ is not but the vertex function, which, after performing the path integrations reads

$$\begin{aligned}\mathcal{B}_\mu(p,p')&=\beta_\mu\exp\{\frac{-ie^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(p.k)(p'.k)}p_\mu D_F^{\mu\nu}(k)p'_\nu\\&\times[(e^{-i\frac{x_0}{p_0}k.p}-1)(1-e^{i\frac{z_0}{p'_0}k.p'})+(e^{i\frac{x_0}{p_0}k.p}-1)(1-e^{-i\frac{z_0}{p'_0}k.p'})]\} ,\end{aligned}\qquad (48)$$

where we have used (36) again. In the asymptotic limit $z_0\to-\infty$, $x_0\to\infty$ the path-ordered phase factor in (48) reduces to

$$\exp\{ie^2\int\frac{d^4k}{(2\pi)^4}\frac{p.p'}{(k^2+i\epsilon)(k.p)(k.p')}\} \qquad (49)$$

in the Feynman gauge. This coincides with the infrared singular exponential which factorizes in the vertex function for both spinorial and the usual scalar QED. In the former it is obtained via perturbation theory[3], while in the later through an alternative laborious method[12].

## 5. Concluding Remarks

We have obtained the meson propagator and the vertex function for the linearized version of scalar QED in the infrared approximation, using the Kubo's technique. In order to make it applicable, the effective interaction Hamiltonian was written as a product of the mesonic current and the electromagnetic field, thus eliminating the quartic interaction term in the S-matrix of the original theory. This also enables us to investigate the causal structure of the S-matrix[14] in the effective theory, and the infrared problem in the adiabatic limit[15] as well.

We have shown that the infrared singularities can be extracted as exponential phase factors from the two- and three-point Green's functions in momentum space. The infrared phase factor (38) in the meson propagator coincides with the divergent exponential factor in the fermion propagator in the Kulish-Faddeev model[6] for $y_0 \to -\infty$, $x_0 \to \infty$. On the other hand, at asymptotic times, the path-ordered phase factor (49) in the Fourier transform of

$$G_\mu(x,0,z) =< 0|T\{\psi(x) : \overline{\psi}(0)\beta_\mu\psi(0) : \overline{\psi}(z)|0 >$$

also coincides with the resulting exponentiation of infrared divergences in the Grammer-Yennie perturbation theory. Therefore, we are led to conclude that spin effects do not manifest in the scattering amplitudes in the infrared region[5],[13], as expression (29) for the asymptotic current of charged particles suggests.

In addition, equation (41) permits one to extend the analisis of infrared divergences in dimensions other than three[11]. This requires closer examination, to be presented in an opportune occasion. Finally, it is noteworthy the remarkable simplicity of the method, which naturally generalizes to the calculation of n-point Green's functions for interactions of type (17), in contrast with the above mentioned methods.

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